A Note on Mutually Unbiased Unextendible Maximally Entangled Bases in $\mathbb{C}^2\bigotimes\mathbb{C}^3$

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Abstract

We systematically study the construction of mutually unbiased bases in $\mathbb{C}^2\bigotimes\mathbb{C}^3$, such that all the bases are unextendible maximally entangled ones. Necessary conditions of constructing a pair of mutually unbiased unextendible maximally entangled bases in $\mathbb{C}^2\bigotimes\mathbb{C}^3$ are derived. Explicit examples are presented.

Mutually unbiased bases (MUBs) play important roles in many quantum information processing such as quantum state tomography [1, 2, 3], cryptographic protocols [4, 5], and the mean kings problem [6]. They are also useful in the construction of generalized Bell states. Let $\mathcal{B}_1=\{|\phi_i\rangle\}$ and $\mathcal{B}_2=\{|\psi_i\rangle\}$, $i=1,2,\cdots,d$, be two orthonormal bases of a d-dimensional complex vector space $\mathbb{C}^d$, $\langle\phi_j|\phi_i\rangle=\delta_{ij}$, $\langle\psi_j|\psi_i\rangle=\delta_{ij}$. $\mathcal{B}_1$ and $\mathcal{B}_2$ are said to be mutually unbiased if and only if

$$|\langle\phi_i|\psi_j\rangle|=\frac{1}{\sqrt{d}}\quad\forall\ i,j=1,2,\cdots,d. \tag{1}$$

Physically if a system is prepared in an eigenstate of basis $\mathcal{B}_1$ and is measured in basis $\mathcal{B}_2$, then all the measurement outcomes have the same probability.

A set of orthonormal bases $\{\mathcal{B}_1,\mathcal{B}_2,...,\mathcal{B}_m\}$ in $\mathbb{C}^d$ is called a set of MUBs if every pair of bases in the set is mutually unbiased. For given dimensional d, the maximum number of MUBs is no more than $d+1$. It has been shown that there are $d+1$ MUBs when d is a prime power [1, 7, 8]. However, for general d, e.g. $d=6$, it is a formidable problem to determine the maximal numbers of MUBs [9, 10, 11, 12, 13, 14, 15, 16, 17].

When the vector space is a bipartite system $\mathbb{C}^d \bigotimes \mathbb{C}^{d'}$ of composite dimension dd', there are different kinds of bases in $\mathbb{C}^d \bigotimes \mathbb{C}^{d'}$ according to the entanglement of the basis vectors. The unextendible product basis (UPB) is a set of incomplete orthonormal product basis whose complementary space has no product states [18]. It is shown that the mixed state on the subspace complementary to a UPB is a bound entangled state. Moreover, the states comprising a UPB are not distinguishable by local measurements and classical communication.

The unextendible maximally entangled basis (UMEB) is a set of orthonormal maximally entangled states in $\mathbb{C}^d \bigotimes \mathbb{C}^d$ consisting of less than d^2 vectors which have no additional maximally entangled vectors that are orthogonal to all of them [19]. Recently, the UMEB in arbitrary bipartite spaces $\mathbb{C}^d \bigotimes \mathbb{C}^{d'}$ has be investigated in [20]. A systematic way in constructing d^2-member UMEBs in $\mathbb{C}^d \bigotimes \mathbb{C}^{d'}$ ($\frac{d'}{2} < d < d'$) is presented. It is shown that the subspace complementary to the d^2-member UMEB contains no states of Schmidt rank higher than $d-1$. From the approach of constructing UMEBs in $\mathbb{C}^d \bigotimes \mathbb{C}^{d'}$, two mutually unbiased UMEBs in $\mathbb{C}^2 \bigotimes \mathbb{C}^3$ are constructed in [20].

In this note, we systematically study the UMEBs in $\mathbb{C}^2 \bigotimes \mathbb{C}^3$ and present a generic way in constructing a pair of UMEBs in $\mathbb{C}^2 \bigotimes \mathbb{C}^3$ such that they are mutually unbiased. The special example given in [20] can be easily obtained from our approach.

A set of states $\{|\phi_i\rangle\}$ in $\mathbb{C}^d \bigotimes \mathbb{C}^{d'}$, $i = 1, 2, \cdots n$, $n < dd'$, is called an n-member UMEB if and only if

(i) all the states $|\phi_i\rangle$ are maximally entangled;

(ii) $\langle\phi_i|\phi_j\rangle = \delta_{i,j}$;

(iii) if $\langle\phi_i|\psi\rangle = 0$, $\forall\ i = 1, 2, \cdots, n$, then $|\psi\rangle$ cannot be maximally entangled.

Here a state $|\psi\rangle$ is said to be a $\mathbb{C}^d \bigotimes \mathbb{C}^{d'}$ maximally entangled state if and only if for an arbitrary given orthonormal complete basis $\{|i_A\rangle\}$ of the subsystem A, there exist an orthonormal basis $\{|i_B\rangle\}$ of the subsystem B such that $|\psi\rangle$ can be written as $|\psi\rangle = \frac{1}{\sqrt{d}}\Sigma_{i=0}^{d-1}|i_A\rangle \otimes |i_B\rangle$ [21].

Let $\{|0\rangle, |1\rangle\}$ and $\{|0'\rangle, |1'\rangle\ |2'\rangle\}$ be the computational bases in $\mathbb{C}^2$ and $\mathbb{C}^3$ respectively. To construct a pair of MUBs which are both UMEBs in $\mathbb{C}^2 \bigotimes \mathbb{C}^3$, we start with the first

UMEB in $\mathbb{C}^2 \bigotimes \mathbb{C}^3$ given by

$$\begin{aligned}
|\phi_i\rangle &= \frac{1}{\sqrt{2}}(\sigma_i \otimes I_3)(|00'\rangle + |11'\rangle),\\
|\phi_4\rangle &= |0\rangle \otimes |2'\rangle,\\
|\phi_5\rangle &= |1\rangle \otimes |2'\rangle,
\end{aligned} \tag{2}$$

where σ_0 denotes the 2×2 identity matrix, σ_i, $i = 1, 2, 3$, are the Pauli matrices, I_3 stands for the 3×3 identity matrix, $|\alpha\beta\rangle \equiv |\alpha\rangle \otimes |\beta\rangle$.

If we choose $\{|a\rangle, |b\rangle\}$ and $\{|x'\rangle, |y'\rangle, |z'\rangle\}$ to be another two bases of $\mathbb{C}^2$ and $\mathbb{C}^3$ respectively, then we have the second UMEB in $\mathbb{C}^2 \bigotimes \mathbb{C}^3$,

$$\begin{aligned}
|\psi_i\rangle &= \frac{1}{\sqrt{2}}(\sigma_i \otimes I_3)(|0x'\rangle + |1y'\rangle),\\
|\psi_4\rangle &= |a\rangle \otimes |z'\rangle,\\
|\psi_5\rangle &= |b\rangle \otimes |z'\rangle.
\end{aligned} \tag{3}$$

The bases $\{|\phi_i\rangle\}$ and $\{|\psi_i\rangle\}$ are mutually unbiased if and only if they satisfy the relations (1),

$$|\langle\phi_i|\psi_j\rangle| = \frac{1}{\sqrt{6}}, \quad \forall\ i, j = 0, 1, \cdots, 5. \tag{4}$$

Let S and W be the unitary matrixes that transforms the bases $\{|0\rangle, |1\rangle\}$ and $\{|0'\rangle, |1'\rangle\ |2'\rangle\}$ to $\{|a\rangle, |b\rangle\}$ and $\{|x'\rangle, |y'\rangle, |z'\rangle\}$ respectively,

$$\begin{aligned}
&S(|0\rangle, |1\rangle) = (|a\rangle, |b\rangle),\\
&W(|0'\rangle, |1'\rangle, |2'\rangle) = (|x'\rangle, |y'\rangle, |z'\rangle).
\end{aligned} \tag{5}$$

Correspondingly we have the relations between $|\phi_i\rangle$ and $|\psi_j\rangle$,

$$\begin{aligned}
|\psi_j\rangle &= (I_2 \otimes W)|\phi_j\rangle, \quad \forall\ j = 0, 1, 2, 3,\\
|\psi_j\rangle &= (S \otimes W)|\phi_j\rangle, \quad \forall\ j = 4, 5.
\end{aligned} \tag{6}$$

From (4) one gets,

$$\begin{aligned}
&|\langle\phi_i|I_2 \otimes W|\phi_j\rangle| = \tfrac{1}{\sqrt{6}},\ \forall\ i = 0, 1, ..., 5,\ \ j = 0, 1, 2, 3,\\
&|\langle\phi_i|S \otimes W|\phi_j\rangle| = \tfrac{1}{\sqrt{6}},\ \forall\ i = 0, 1, ..., 5,\ \ j = 4, 5.
\end{aligned} \tag{7}$$

As $\{|\phi_i\rangle\}$ forms a base in $\mathbb{C}^2 \bigotimes \mathbb{C}^3$, the relations in (7) imply that the absolute values of the entries of the matrices $I \otimes W$ and $S \otimes W$ under the base $\{|\phi_i\rangle\}$ have the following forms:

$$\begin{pmatrix} \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & X & X \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & X & X \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & X & X \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & X & X \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & X & X \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & X & X \end{pmatrix}, \tag{8}$$

$$\begin{pmatrix} X & X & X & X & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ X & X & X & X & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ X & X & X & X & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ X & X & X & X & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ X & X & X & X & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \\ X & X & X & X & \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} \end{pmatrix}, \tag{9}$$

where X denotes any numbers.

Let

$$S = \begin{pmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{pmatrix}, \quad W = \begin{pmatrix} w_{11} & w_{12} & w_{13} \\ w_{21} & w_{22} & w_{23} \\ w_{31} & w_{32} & w_{33} \end{pmatrix} \tag{10}$$

be the matrices of S and W in the computational product basis $\{|0\rangle, |1\rangle\} \otimes \{|0'\rangle, |1'\rangle, |2'\rangle\}$. Let F be the unitary matrix that transforms the computational product basis to the basis $\{|\phi_i\rangle\}$, i.e.,$F(|00'\rangle, |01'\rangle, |02'\rangle, |10'\rangle, |11'\rangle, |12'\rangle) = (|\phi_0\rangle, ..., |\phi_5\rangle)$. Form (2), one can easily get

$$F = \begin{pmatrix} \frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\ \frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}. \tag{11}$$

Therefore the matrices of $I_2 \otimes W$ and $S \otimes W$ under the basis $\{|\phi_i\rangle\}$ are given by

$$F^{\dagger}(I_2 \otimes W)F, \tag{12}$$

and

$$F^{\dagger}(S \otimes W)F, \tag{13}$$

respectively.

Comparing (12) and (13) with (8) and (9), by straightforward calculations, we have

(i) The absolute values of the entries of w are $1/\sqrt{3}$. Moreover, in the complex plane, $w_{11} \perp w_{22}$ and $w_{21} \perp w_{12}$.

(ii) The absolute values of the entries of S is $1/\sqrt{2}$. In the complex plane, $w_{13}s_{11} \perp w_{23}s_{21}$, $w_{23}s_{11} \perp w_{13}s_{21}$, $w_{13}s_{12} \perp w_{23}s_{22}$ and $w_{23}s_{12} \perp w_{13}s_{22}$.

From the condition (i), for simplification, we can set

$$W = 1/\sqrt{3}\begin{pmatrix} e^{i\theta_1} & e^{i(\theta_2+\frac{\pi}{2})} & e^{i\theta_4} \\ e^{i\theta_2} & e^{i(\theta_1+\frac{\pi}{2})} & e^{i\theta_5} \\ e^{i\theta_3} & e^{i(\theta_3-\frac{\pi}{2})} & e^{i\theta_6} \end{pmatrix}, \tag{14}$$

where, due the properties of unitary matrix, θ_i satisfy the following conditions,

$$\begin{aligned} &|\theta_1-\theta_2| = \tfrac{\pi}{3}, \quad |\theta_4-\theta_5| = \pi, \\ &e^{i(\theta_1-\theta_4)}e^{-i\pi/3} + e^{i(\theta_3-\theta_6)} = 0. \end{aligned} \tag{15}$$

From equation (14), (15) and condition (ii), we find s_{11} and s_{21} are orthogonal, s_{12} and s_{22} are orthogonal. Then we can simply set

$$S = \frac{1}{\sqrt{2}}\begin{pmatrix} e^{i\theta_1'} & e^{i\theta_2'} \\ \pm e^{i(\theta_1'+\frac{\pi}{2})} & \mp e^{i(\theta_2'+\frac{\pi}{2})} \end{pmatrix}. \tag{16}$$

where θ_1', θ_2' can be any real numbers.

Therefore, for any θ_is and θ_i's satisfying (15) and (16) respectively, one has a W and a S. Then from (6) one gets the UMEB $\{|\psi_i\rangle\}$ that is mutually unbiased with the UMEB $\{|\phi_i\rangle\}$.

We next give some concrete examples of mutually unbiased UMEBs in $\mathbb{C}^2 \bigotimes \mathbb{C}^3$.

The UMEB $\{|\phi_i\rangle\}$ presented in [20] is of the form,

$$\begin{aligned}
|\phi_0\rangle &= \frac{1}{\sqrt{2}}(|00'\rangle + |11'\rangle), \\
|\phi_i\rangle &= \frac{1}{\sqrt{2}}(\sigma_i \otimes I_3)(|00'\rangle + |11'\rangle), \quad i = 1,2,3, \\
|\phi_4\rangle &= |c\rangle \otimes |2'\rangle, \\
|\phi_5\rangle &= |d\rangle \otimes |2'\rangle,
\end{aligned} \tag{17}$$

where $|c\rangle = \frac{1}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle, |d\rangle = \frac{\sqrt{3}}{2}|0\rangle - \frac{1}{2}|1\rangle$. This example corresponds to a different transformation matrix F,

$$F = \begin{pmatrix}
\frac{1}{\sqrt{2}} & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\
0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{1}{2} & \frac{\sqrt{3}}{2} \\
0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 & 0 \\
\frac{1}{\sqrt{2}} & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 \\
0 & 0 & 0 & 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2}
\end{pmatrix}.$$

From our approach, $w_{31} \perp w_{32}$ should be added to the condition (i). With respect to the condition (ii), the orthogonal relation becomes $(s_{11} + \sqrt{3}s_{12}) \perp (s_{21} + \sqrt{3}s_{22})$ and $(\sqrt{3}s_{11} - s_{12}) \perp (\sqrt{3}s_{21} - s_{22})$. However, since we have already set $w_{31} \perp w_{32}$ in (14), (15) can be also used for this example.

We choose $\{\theta_i\}$ to be

$$\{\theta_1 = 0,\, \theta_2 = \frac{\pi}{3},\, \theta_3 = 0, \theta_4 = \pi, \theta_5 = 0,\, \theta_6 = \frac{\pi}{3}\}, \tag{18}$$

which satisfy the condition (15). From (14) we have

$$W = 1/\sqrt{3} \begin{pmatrix}
1 & \frac{-\sqrt{3}+i}{2} & -1 \\
\frac{1+\sqrt{3}i}{2} & i & 1 \\
1 & -i & \frac{1+\sqrt{3}i}{2}
\end{pmatrix}. \tag{19}$$

The unitary matrix W transforms the basis $\{|0'\rangle, |1'\rangle\, |2'\rangle\}$ to basis $\{|x'\rangle, |y'\rangle, |z'\rangle\}$.

From (5) we have

$$
\begin{aligned}
|x'\rangle &= \frac{1}{\sqrt{3}}(|0'\rangle + \frac{1+\sqrt{3}i}{2}|1'\rangle + |2'\rangle),\\
|y'\rangle &= \frac{1}{\sqrt{3}}(\frac{-\sqrt{3}+i}{2}|0'\rangle + i|1'\rangle - i|2'\rangle),\\
|z'\rangle &= \frac{1}{\sqrt{3}}(-|0'\rangle + |1'\rangle + \frac{1+\sqrt{3}i}{2}|2'\rangle).
\end{aligned} \tag{20}
$$

We have the unitary operator S,

$$
S = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ \frac{\sqrt{3}+i}{2} & \frac{1-\sqrt{3}\,i}{2} \end{pmatrix}. \tag{21}
$$

The corresponding operator S, $S(|c\rangle, |d\rangle) = (|a\rangle, |b\rangle)$, give rise to

$$
\begin{aligned}
|a\rangle &= \frac{1}{\sqrt{2}}(\frac{1+\sqrt{3}i}{2}|0\rangle + \frac{\sqrt{3}-i}{2}|1\rangle),\\
|b\rangle &= \frac{1}{\sqrt{2}}(\frac{\sqrt{3}-i}{2}|0\rangle + \frac{1+\sqrt{3}i}{2}|1\rangle).
\end{aligned} \tag{22}
$$

Therefore, the second UMEB that is mutually unbiased to (17) is given by

$$
\begin{aligned}
|\psi_j\rangle &= \frac{1}{\sqrt{2}}(\sigma_i \otimes I_3)(|0x'\rangle + |1y'\rangle), \quad j = 1, 2, 3,\\
|\psi_4\rangle &= \frac{1}{\sqrt{2}}(\frac{1+\sqrt{3}i}{2}|0\rangle + \frac{\sqrt{3}-i}{2}|1\rangle) \otimes |z'\rangle,\\
|\psi_5\rangle &= \frac{1}{\sqrt{2}}(\frac{\sqrt{3}-i}{2}|0\rangle + \frac{1+\sqrt{3}i}{2}|1\rangle) \otimes |z'\rangle.
\end{aligned} \tag{23}
$$

(17) and (23) are exactly the ones presented in [20].

Now we give a new example by choosing other values of $\{\theta_i\}$ and $\{\theta_i'\}$. Let the first UMEB in $\mathbb{C}^2 \bigotimes \mathbb{C}^3$ be the one given in (2). Taking into the condition (15), we set

$$
\theta_1 = \pi,\ \theta_2 = \frac{2\pi}{3},\ \theta_3 = \theta_4 = 0,\ \theta_5 = \pi,\ \theta_6 = \frac{\pi}{3}. \tag{24}
$$

From (14), we get

$$
W = 1/\sqrt{3} \begin{pmatrix} -1 & \frac{-\sqrt{3}-i}{2} & 1 \\ \frac{-1+\sqrt{3}i}{2} & -i & -1 \\ 1 & -i & \frac{1+\sqrt{3}i}{2} \end{pmatrix}, \tag{25}
$$

and

$$
\begin{aligned}
|x'\rangle &= \frac{1}{\sqrt{3}}(-|0'\rangle + \frac{-1+\sqrt{3}i}{2}|1'\rangle + |2'\rangle), \\
|y'\rangle &= \frac{1}{\sqrt{3}}(\frac{-\sqrt{3}-i}{2}|0'\rangle - i|1'\rangle - i|2'\rangle), \\
|z'\rangle &= \frac{1}{\sqrt{3}}(|0'\rangle - |1'\rangle + \frac{1+\sqrt{3}i}{2}|2'\rangle).
\end{aligned}
\tag{26}
$$

Taking $\theta'_1 = 0$ and $\theta'_2 = \frac{\pi}{2}$, we have

$$
S = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix},
\tag{27}
$$

and

$$
|a\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle), \qquad |b\rangle = \frac{1}{\sqrt{2}}(i|0\rangle + |1\rangle).
\tag{28}
$$

From (3) we obtain the second UMEB that is mutually unbiased to the UMEB given by Eq. (2),

$$
\begin{aligned}
|\psi_j\rangle &= \frac{1}{\sqrt{2}}(\sigma_i \otimes I_3)(|0x'\rangle + |1y'\rangle), \quad j = 1,2,3, \\
|\psi_4\rangle &= \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle) \otimes |z'\rangle, \\
|\psi_5\rangle &= \frac{1}{\sqrt{2}}(i|0\rangle + |1\rangle) \otimes |z'\rangle.
\end{aligned}
\tag{29}
$$

It can be directly verified that the two UMEBs (17) and (29) satisfy the condition (4).

As another example we choose

$$
\begin{aligned}
&\theta_1 = \frac{4\pi}{3},\ \theta_2 = \pi,\ \theta_3 = 0, \theta_4 = \pi, \theta_5 = 0, \theta_6 = \pi, \\
&\theta'_1 = \frac{\pi}{3}, \theta'_2 = \frac{\pi}{6}.
\end{aligned}
\tag{30}
$$

The corresponding unitary matrix W and S are of the form,

$$
W = 1/\sqrt{3} \begin{pmatrix} \frac{-1-\sqrt{3}i}{2} & -i & -1 \\ -1 & \frac{\sqrt{3}-i}{2} & 1 \\ 1 & -i & -1 \end{pmatrix},
\tag{31}
$$

$$
S = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1+\sqrt{3}i}{2} & \frac{\sqrt{3}+i}{2} \\ \frac{-\sqrt{3}+i}{2} & \frac{1-\sqrt{3}i}{2} \end{pmatrix}.
\tag{32}
$$

The basis $\{|a\rangle, |b\rangle\}$ in $\mathbb{C}^2$ and the basis $\{|x'\rangle, |y'\rangle, |z'\rangle\}$ in $\mathbb{C}^3$ are given by

$$\begin{aligned}
|x'\rangle &= \frac{1}{\sqrt{3}}(\frac{-1-\sqrt{3}i}{2}|0'\rangle - |1'\rangle + |2'\rangle), \\
|y'\rangle &= \frac{1}{\sqrt{3}}(-i|0'\rangle + \frac{\sqrt{3}-i}{2}|1'\rangle - i|2'\rangle), \\
|z'\rangle &= \frac{1}{\sqrt{3}}(-|0'\rangle + |1'\rangle - |2'\rangle),
\end{aligned}$$

and

$$\begin{aligned}
|a\rangle &= \frac{1}{\sqrt{2}}(\frac{1+\sqrt{3}i}{2}|0\rangle + \frac{-\sqrt{3}+i}{2}|1\rangle), \\
|b\rangle &= \frac{1}{\sqrt{2}}(\frac{\sqrt{3}+i}{2}|0\rangle + \frac{1-\sqrt{3}i}{2}|1\rangle).
\end{aligned} \tag{33}$$

Therefore, another UMEB that is mutually unbiased to the UMEB given by (2) is of the form,

$$\begin{aligned}
|\psi_j\rangle &= \frac{1}{\sqrt{2}}(\sigma_i \otimes I_3)(|0x'\rangle + |1y'\rangle), \quad j = 1, 2, 3, \\
|\psi_4\rangle &= \frac{1}{\sqrt{2}}(\frac{1+\sqrt{3}i}{2}|0\rangle + \frac{-\sqrt{3}+i}{2}|1\rangle) \otimes |z'\rangle, \\
|\psi_5\rangle &= \frac{1}{\sqrt{2}}(\frac{\sqrt{3}+i}{2}|0\rangle + \frac{1-\sqrt{3}i}{2}|1\rangle) \otimes |z'\rangle.
\end{aligned} \tag{34}$$

We have presented a general way in constructing UMEBs in $\mathbb{C}^2 \bigotimes \mathbb{C}^3$ such that they are mutually unbiased. Explicit examples are given for constructing a pair of mutually unbiased unextendible maximally entangled bases, including the one in [20] as a special case. Our approach may shed light in constructing more UMEBs that are pairwise mutually unbiased in $\mathbb{C}^2 \bigotimes \mathbb{C}^3$ or higher dimensional bipartite systems.

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